Local and Global Topology Preservation
in Locally Finite Sets of Tiles

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Abstract

This paper deals with sets $\mathcal{P}$ of *tiles* (compact, convex sets) in R^n. Tiles are a generalization of pixels or voxels (in R^2 or R^3); they can have arbitrary shapes and are allowed to overlap. The union of all the tiles of $\mathcal{P}$ is denoted by $\mathcal{U}(\mathcal{P})$. The *neighborhood* $N_{\mathcal{P}}(P)$ of a tile P is the union of the tiles of $\mathcal{P}$ that intersect $\mathcal{P}$. P is called *simple* if deletion of P from $\mathcal{P}$ does not change the topology (in the homotopy sense) of $\mathcal{U}(\mathcal{P})$. We show in this paper that if $\mathcal{P}$ satisfies a property called *strong normality* (SN), and deletion of P preserves the topology of $N_{\mathcal{P}}(P)$, then P is simple. This may not be true if $\mathcal{P}$ is not SN; and even if $\mathcal{P}$ is SN, P may be simple even if deletion of P does not preserve the topology of $N_{\mathcal{P}}(P)$.

Keywords: Digital topology, simple voxel, local topology, global topology

1 Introduction

Let $\mathcal{P}$ be a set of compact (closed and bounded) convex sets in R^n; the elements of $\mathcal{P}$ (which correspond to pixels or voxels in conventional two- or three-dimensional digital images) will be called *tiles*, and the union of all the elements of $\mathcal{P}$ will be denoted by $\mathcal{U}(\mathcal{P})$. $\mathcal{P}$ is called *locally finite* if, for any $P \in \mathcal{P}$, the number of tiles that intersect P is finite. $\mathcal{P}$ will be called *strongly normal* (SN) if it is locally finite and for all $P, P_1, P_2, \cdots, P_n (n \geq 1) \in \mathcal{P}$, if each P_i intersects P and $I = P_1 \cap P_2 \cap \cdots \cap P_n$ is nonempty, then I intersects P. It is not difficult to see that strong normality is hereditary: If it holds for $\mathcal{P}$, it holds for every $\mathcal{P}' \subseteq \mathcal{P}$.

The *neighborhood* of P in $\mathcal{P}$, denoted by $N_{\mathcal{P}}(P)$, is the union of all $Q \in \mathcal{P}$ that intersect P (including P itself). In [1] it is proved, for $n = 3$, that if $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$ the neighborhood $N_{\mathcal{P}'}(P)$ of any $P \in \mathcal{P}'$ is simply connected—i.e., it cannot have a tunnel—and has no cavities. The *deleted neighborhood* of P in $\mathcal{P}$, denoted by $N_{\mathcal{P}'}^*(P)$, is the union of all $Q \neq P \in \mathcal{P}$ that intersect P.

$P \in \mathcal{P}$ is called *simple* if deletion of P from $\mathcal{P}$ does not change the topology (in the homotopy sense) of $\mathcal{U}(\mathcal{P})$. In Section 2 of this paper we show that if $\mathcal{P}$ is SN and $N_{\mathcal{P}}^*(P)$ is simply connected and without cavities, so that deletion of P from $\mathcal{P}$ preserves the topology of $N_{\mathcal{P}}(P)$, then P is simple. Thus in an SN set of tiles, preservation of local topology when P is deleted is a sufficient condition for simplicity of P. (This need not be true in a non-SN locally finite set of tiles, as we show in Section 3.) On the other hand, we show in Section 4 that even in an SN set of tiles, local topology preservation is not a necessary condition for simplicity; if $\mathcal{P}$ satisfies certain not very restrictive conditions, a tile may be simple even if its deletion fails to preserve topology in a "neighborhood" of any fixed size; thus preservation of local topology when P is deleted is not a necessary condition for simplicity of P.

2 In an SN set of tiles, if deletion of a tile preserves local topology, it preserves global topology

For any $\Delta > 0$ and any subset S of R^n, the Δ-*closure* of a subset X of S in S, denoted by closure(X, S, Δ), is defined as the set $\{p | p \in S \text{ and } \delta(p, X) \leq \Delta\}$, where $\delta(p, X)$ is the Euclidean distance between p and the nearest point of X. The *Hausdorff distance* $\delta_H(S_1, S_2)$ [2] between two sets $S_1, S_2 \subset R^n$ is defined as $\max(\max_{p \in S_1} \delta(p, S_2), \max_{p \in S_2} \delta(p, S_2))$. Let $\Pi(S)$ denote the power set of S. A *continuous deformation* σ in S from A to B (where $A, B \subseteq S$) is a mapping $\sigma : [0, 1] \rightarrow \Pi(S)$ such that $\sigma(0) = A$, $\sigma(1) = B$, and for every $x, y \in [0, 1]$, as y approaches x, $\delta_H(\sigma(x), \sigma(y))$ approaches zero. Such a deformation will be called *confined* to X if, for every $x, y \in [0, 1]$, $(\sigma(x) - \sigma(y)) \cup (\sigma(y) - \sigma(x))$ is a subset of X. It will be called *topology-preserving* if, for all $x, y \in [0, 1]$, as y approaches x, for all $p \in \sigma(x) - \sigma(y)$ there exists a Δ such that, for all $\Delta' \leq \Delta$, both closure $(p, \sigma(x), \Delta')$ and closure$(p, \sigma(y), \Delta')$ are simply connected and without cavities.

Theorem 1 *If $\mathcal{P}$ is SN, then for any $\mathcal{P}' \subseteq \mathcal{P}$, if $P \in \mathcal{P}'$ and $N_{\mathcal{P}'}^*(P)$ is simply connected and without cavities, the deletion of P from $\mathcal{P}'$ does not change the topology (in the homotopy sense) of $\mathcal{U}(\mathcal{P}')$.*

Proof: $N_{\mathcal{P}'}^*(P)$ is simply connected without cavities, and it is contained in $N_{\mathcal{P}'}(P)$, which by SN is also simply connected and without cavities. Hence [3] there exists a topology-preserving deformation σ (a deformation retraction), confined to $N_{\mathcal{P}'}(P) - N_{\mathcal{P}'}^*(P)$, that takes $N_{\mathcal{P}'}(P)$ into $N_{\mathcal{P}'}^*(P)$. (Note that P may be contained in $N_{\mathcal{P}'}^*(P)$, so that $N_{\mathcal{P}'}(P)$ and $N_{\mathcal{P}'}^*(P)$ are the same, in which case σ is the identity mapping.) Let σ' be the mapping defined by $\sigma'(x) = \sigma(x) \cup (\mathcal{U}(\mathcal{P}') - N_{\mathcal{P}'}(P))$. It is not difficult to see that σ' is a continuous deformation in $\mathcal{U}(\mathcal{P}')$ from $\mathcal{U}(\mathcal{P}')$ to $\mathcal{U}(\mathcal{P}' - \{P\})$ that is confined to $N_{\mathcal{P}'}(P) - N_{\mathcal{P}'}^*(P)$. Suppose σ' were not topology-preserving; then there would exist some x, y, in [0,1] such that as y approaches x, there always exists a point p in $\sigma'(x) - \sigma'(y)$ such that for every sufficiently small Δ, one or both of closure$(p, \sigma'(x), \Delta)$ and closure$(p, \sigma'(y), \Delta)$ is not simply connected and without cavities. But $\sigma'(x) - \sigma'(y) \subseteq \sigma(x) - \sigma(y)$, so that $p \in \sigma(x) - \sigma(y)$. Since σ is confined to $N_{\mathcal{P}'}(P) - N_{\mathcal{P}'}^*(P)$, we thus have $p \in P$. Since the Ps are compact subsets of R^n, for a sufficiently small Δ, closure$(p, \mathcal{U}(\mathcal{P}') - N_{\mathcal{P}'}(P), \Delta)$ is empty, so that closure$(p, \sigma'(x), \Delta) = $ closure$(p, \sigma(x), \Delta)$ and closure$(p, \sigma'(y), \Delta) = $ closure$(p, \sigma(y), \Delta)$; this contradicts the fact that σ is topology-preserving. $\qquad\square$

3 This need not be true in a locally finite set of tiles that is not SN

In this section we prove that the following inverse of Theorem 1 is also true: If $\mathcal{P}$ is a locally finite set of tiles that violates SN, then there exist $\mathcal{P}' \subseteq \mathcal{P}$ and $P \in \mathcal{P}'$ such that $N_{\mathcal{P}'}^*(P)$ is simply connected without cavities, but the removal of P fails to preserve the topology (in the homotopy sense) of $\mathcal{U}(\mathcal{P}')$.

Lemma 1 *Let P be a tile in a locally finite set of tiles $\mathcal{P}$. If $Q_1, Q_2, \cdots, Q_n$ is a minimal set of neighbors of P in $\mathcal{P}$ that violates SN, then n is either 2 or 3.*

Proof. See [1]. $\qquad\square$

If $k = 2$, $P \cap Q_1$ and $P \cap Q_2$ must be disjoint. If $k = 3$, $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$ must be nonempty and disjoint.

Theorem 2 *Let $\mathcal{P}$ be a locally finite set of tiles that violates SN; then there exist $\mathcal{P}' \subseteq \mathcal{P}$ and $P \in \mathcal{P}'$ such that $N_{\mathcal{P}'}^*(P)$ is simply connected without cavities, but the removal of P fails to preserve the topology (in the homotopy sense) of $\mathcal{U}(\mathcal{P}')$.*

Proof: By Lemma 1, since $\mathcal{P}$ is not SN, there exists a set of two or three Q's (neighbors of some $P \in \mathcal{P}$) that violates SN.

Suppose first that this set has two elements Q_1, Q_2; let $\mathcal{P}' = \{P, Q_1, Q_2\}$. Let C be a closed curve in $N_{\mathcal{P}'}(P) = P \cup Q_1 \cup Q_2$ that passes through each of the intersections $P \cap Q_1$, $P \cap Q_2$ and $Q_1 \cap Q_2$. It can be shown (see [1] for the details) that in $N_{\mathcal{P}'}(P)$, C is not reducible to a point; in other words, $N_{\mathcal{P}'}(P) = P \cup Q_1 \cup Q_2$ has a tunnel and so is not simply connected. Let $\mathcal{P}'' = \{Q_1, Q_2\}$. $\mathcal{P}''$ is (trivially) SN, so that by [1], $N_{\mathcal{P}''}(Q_1) = Q_1 \cup Q_2$ is simply connected and without cavities; hence $N_{\mathcal{P}'}^*(P) = Q_1 \cup Q_2$ is simply connected and without cavities. Thus $\mathcal{U}(\mathcal{P}') = N_{\mathcal{P}'}(P)$ is not simply connected, while $\mathcal{U}(\mathcal{P}' - \{P\}) = N_{\mathcal{P}'}^*(P)$

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is simply connected and without cavities. Hence the removal of P from $\mathcal{P}'$ fails to preserve the topology of $\mathcal{U}(\mathcal{P}')$.

Next, suppose the minimal set of Q's that violates SN has three elements Q_1, Q_2, Q_3. Let $\mathcal{P}' = \{P, Q_1, Q_2, Q_3\}$; since the Q's violate SN, their intersection must be nonempty, and $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$ must also be nonempty. Let p, p_3, p_1 and p_2 be points in $Q_1 \cap Q_2 \cap Q_3$, $P \cap Q_1 \cap Q_2$, $P \cap Q_2 \cap Q_3$, and $P \cap Q_3 \cap Q_1$, respectively, such that the volume of the tetrahedron T defined by p, p_3, p_1 and p_2 is minimum. It can be shown (see [1] for the details) that $N_{\mathcal{P}'}(P)$ fails to occupy the entire interior of T, and that the interior is surrounded by $N_{\mathcal{P}'}(P)$; hence $N_{\mathcal{P}'}(P)$ has a cavity. Let $\mathcal{P}'' = \{Q_1, Q_2, Q_3\}$. Then $\mathcal{P}''$ is SN, so that by [1], $N_{\mathcal{P}''}(Q_1) = Q_1 \cup Q_2 \cup Q_3$ is simply connected and without cavities; hence $N_{\mathcal{P}'}^*(P) = Q_1 \cup Q_2 \cup Q_3$ is simply connected and without cavities. But $\mathcal{U}(\mathcal{P}') = N_{\mathcal{P}'}(P)$ has a cavity, while $\mathcal{U}(\mathcal{P}' - \{P\}) = N_{\mathcal{P}'}^*(P)$ is simply connected and without cavities. Hence the removal of P from $\mathcal{P}'$ fails to preserve the topology of $\mathcal{U}(\mathcal{P}')$ □

Note that in Theorem 2, removal of P does not even preserve local topology, because $N_{\mathcal{P}'}(P)$ has a tunnel or a cavity, while $N_{\mathcal{P}'}^*(P)$ is simply connected and without cavities. Figure 1 shows an example in which both $N_{\mathcal{P}}(P)$ and $N_{\mathcal{P}}^*(P)$ are connected and have one tunnel and no cavities (note that R is not in $N_{\mathcal{P}}(P)$), so that local topology is preserved by removal of P, but global topology is not: $\mathcal{U}(P)$ has a tunnel, which is destroyed when P is removed.

4   Deletion of a tile may preserve global topology even if it does not preserve local topology

In this section we show that if $\mathcal{P}$ is SN and satisfies certain simple conditions, there can exist tiles in $\mathcal{P}$ whose deletion preserves global topology, but does not preserve local topology in a neighborhood of any fixed order. [For any natural number n and any tile P, we define the nth-order neighborhood $N_{\mathcal{P}}^n(P)$ of P inductively by: $N_{\mathcal{P}}^0(P) = P$; $N_{\mathcal{P}}^n(P) = \{\bigcup(Q \in \mathcal{P}) | Q$ intersects $N_{\mathcal{P}}^{n-1}(P)\}$. We similarly define $N_{\mathcal{P}}^{n*}(P)$ as the union of the tiles in $N_{\mathcal{P}}^n(P)$ except for P itself.] We will show in this section that there can exist tiles in $\mathcal{P}$ such that $N_{\mathcal{P}}^{n*}$ is not topologically equivalent to $N_{\mathcal{P}}^n$. Thus local topology preservation in such a $\mathcal{P}$ is not a necessary condition for global topology preservation.

A tile P will be called *redundant* in $\mathcal{P}$ if $N_{\mathcal{P}}^*(P)$ contains P. $\mathcal{P}$ will be called *irredundant* if no tile of $\mathcal{P}$ is redundant. We assume from now on that $\mathcal{P}$ is irredundant and SN.

Let $Q_1, \ldots, Q_m$ be tiles that intersect P, and let $Q = \bigcup_{i=1}^m Q_i$. Since $\mathcal{P}$ is irredundant, Q does not contain P.

Proposition 1. Q surrounds P iff it contains Boundary(P).

Proof: "If" is clear, since Boundary(P) surrounds P. Conversely, let x be a point of Boundary(P) that is not contained in Q. Since tiles are closed sets, there is a point x' of the complement of P, close to x, that is still not contained in Q and is evidently still surrounded by Q. Thus x' is in the complement of $P \cup Q$ and is surrounded by $P \cup Q$, so that $P \cup Q$ has a cavity; but since $\mathcal{P}' = \{P, Q_1, \ldots, Q_m\}$ is SN, $N_{\mathcal{P}'}(P) = P \cup Q$ cannot have a cavity, contradiction. □

Note that since Q surrounds P but does not contain P, some connected component of Q has a cavity.

3

We say that Q *encircles* P if $P - Q$ is connected but Boundary$(P) - Q$ is not connected. [Note that analogously, in Proposition 1, $P - Q$ is nonempty but Boundary$(P) - Q$ is empty. Evidently if Q surrounds P it cannot encircle P, and vice versa.]

Proposition 2. If Q encircles P, some connected component of Q has a tunnel.

Proof: Let x, y be points belonging to different components of Boundary$(P) - Q$. Then there is a curve c in $Q \cap$ Boundary(P) that separates x and y in Boundary(P). Since $P - Q$ is connected, there is a path p from x to y in $P - Q$. Since tiles are closed sets, there exist points x', y' in the complement of P, close to x and y, that are still not contained in Q. Since $\mathcal{P}' = \{P, Q_1, \ldots, Q_m\}$ is SN, $P \cup Q = N_{\mathcal{P}'}(P)$ has no cavity; hence x' and y' cannot be surrounded by $P \cup Q$ (since they are in the complement of $P \cup Q$). Hence there is a path p' from x to y (via x' and y') in the complement of $P \cup Q$. The concatenation of p and p' is a curve d in the complement of Q. It is not hard to see that c and d are linked; thus Q has a tunnel. $\square$

It is not hard to see that in Proposition 1 or 2, if m is minimal, Q must be connected. Note that since $\mathcal{P}' = \{P, Q_1, \ldots, Q_m\}$ is SN, $N_{\mathcal{P}'}(P) = P \cup Q$ has no tunnel. It is not hard to see that if m is minimal, Boundary$(P) - Q$ has exactly two connected components and Q has just one tunnel.

The *distance* $d(P, Q)$ between two tiles P, Q is $\inf\{d(x, y) | x \in P, y \in Q\}$, where d is Euclidean distance. We call $\mathcal{P}$ *broad* at P if P has at least two disjoint neighbors, and for any two such neighbors U_1, V_1, there exist sequences of tiles $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ such that (1) U_i is never a neighbor of V_j; (2a) U_i is a neighbor of U_j iff $|i - j| = 1$, and V_i is a neighbor of V_j iff $|i - j| = 1$; (2b) if any $U_i(V_i)$ is adjacent to any neighbor Q of P, then $U_1(V_1)$ is also adjacent to Q; (2c) no U_i or V_i is contained in $N_P^{(i-1)}(P)$; (2d) for any distance D, there exist i, j such that $d(P, U_i) > D$ and $d(P, V_j) > D$. (This follows from (2c) if the volumes of the tiles are bounded below.)

We call $\mathcal{P}$ *broadly connected* if, for any P and any D, if $\mathcal{P}'_{D,P}$ is the result of deleting from $\mathcal{P}$ all the tiles whose distances from P are at most D, then $\mathcal{U}(\mathcal{P}'_{D,P})$ is nonempty and connected. Evidently if $\mathcal{P}$ is broadly connected, it is connected and unbounded.

Let $\mathcal{P}$ be broad at P, and for any D, let $U_{i(D)}, V_{j(D)}$ be the first U_i and the first V_j such that $d(P, U_{i(D)}) > D$ and $d(P, V_{j(D)}) > D$. If $\mathcal{P}$ is broadly connected, there is a path in $\mathcal{U}(\mathcal{P}'_{D,P})$ from $U_{i(D)}$ to $V_{j(D)}$. Let p be such a path that passes through as few tiles as possible, say through $U_{i(D)} = W_1, \ldots, W_k = V_{j(D)}$; then W_i and W_j are neighbors iff $|i - j| \leq 1$, and no W is a neighbor of any U_i or V_j for $i < i(D)$ or $j < j(D)$. Thus the cyclic sequence of tiles $P, U_1, \ldots U_{i(D)} = W_1, \ldots, W_k = V_{j(D)}, \ldots, V_1$ is a digital simple closed curve; it is not difficult to show that the union of these tiles is connected, has no cavities, and has exactly one tunnel. Moreover, since U_1 and V_1 are not neighbors of each other, if P is deleted, this curve becomes an arc: it is connected and has no cavities or tunnels.

The *diameter* $\delta(P)$ of a tile P is $\sup\{d(x, y) | x, y \in P\}$, where d is Euclidean distance. We will assume that the tiles of $\mathcal{P}$ have bounded diameters, i.e. that there exists a Δ such that $\delta(P) \leq \Delta$ for all P.

Theorem 3. Let $\mathcal{P}$ be broadly connected and broad at P, and let P have neighbors $Q_1, \ldots, Q_m$ (where m is minimal) such that $Q = \bigcup_{i=1}^m Q_i$ encircles P, as well as another neighbor R disjoint from Q. Then for any $n \geq 1$, deletion of P from $\mathcal{P}'$ does not preserve the topology of $N_{\mathcal{P}'}^n(P)$, but does preserve the topology of $\mathcal{U}(\mathcal{P}')$.

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Proof: Let $\mathcal{P}' = \{P, Q_1, \ldots, Q_m, R = U_1, \ldots, U_{i(D)} = W_1, \ldots, W_k = V_{j(D)}, \ldots, V_1\}$, where V_1 is one of the Q's, and where $D > n\Delta$. Then $N_{\mathcal{P}'}^n(P)$ is the union of $P, Q, U_1, \ldots, U_n$, $V_2, \ldots, V_n$ (note that $V_1 \subseteq Q$), where $n < i(D), j(D)$ since $n\Delta < D$. Before P is deleted, $N_{\mathcal{P}'}^n(P)$ is connected and has no tunnel or cavity. [Since $R = U_1$ is not a neighbor of any of the Q's, by (2b), no U_i can be a neighbor of any of the Q's. By (1), V's are never neighbors of U's, and by (2c), no V_j for $j \geq 3$ can be a neighbor of any Q. Let $\mathcal{P}'' = \{P, Q_1, \ldots, Q_m, V_2\}$, and let $\mathcal{P}'''$ be the set of Q's that are neighbors of V_2. Evidently, $V_1 \in \mathcal{P}'''$, and by (2b), every tile in $\mathcal{P}'''$ is a neighbor of V_1. Hence $N_{\mathcal{P}'''}(V_1) = N_{\mathcal{P}''}^*(V_2)$. But by SN, $N_{\mathcal{P}'''}(V_1)$ is simply connected and without cavities; hence $N_{\mathcal{P}''}^*(V_2)$ is simply connected and without cavities.]

After P is deleted, $N_{\mathcal{P}'}^{n*}(P)$ is not connected (the U's are not connected to the Q's, although the V's may be). Moreover, $N_{\mathcal{P}'}^{n*}(P)$ has a tunnel, since Q has a tunnel, and none of the tiles in $N_{\mathcal{P}'}^{n*}(P)$ can block this tunnel. [This is clear except possibly for V_2. If a union of tiles $B_1, \ldots, B_k$ blocks the tunnel in Q, every Q_j must be a neighbor of some B_i (otherwise there would exist a curve arbitrarily close to Q_j that avoids the B's and is linked with the curve contained in Q). Thus the only tile in $N_{\mathcal{P}'}^{n*}(P)$ that can block the tunnel in Q is V_2, and if it does so, it must be a neighbor of every Q_j. But by (2b), this implies that V_1 is also a neighbor of every Q_j, so that if we let $\mathcal{P}'' = \{Q_1, \ldots, Q_m\}$, then $N_{\mathcal{P}''}(V_1) = Q$ has a tunnel; but by SN, $N_{\mathcal{P}''}(V_1)$ must be simply connected, contradiction.] Thus deleting P does not preserve the topology of $N_{\mathcal{P}'}^n(P)$. On the other hand, $\mathcal{U}(\mathcal{P}')$ is connected and has a tunnel (but no cavity). When P is deleted, $\mathcal{U}(\mathcal{P}' - \{P\})$ is still connected (through the W's), and although it no longer has the tunnel defined by the digital simple closed curve, it now has the tunnel in Q and it still has no cavity. Thus deleting P does preserve the topology of $\mathcal{U}(\mathcal{P}')$. $\square$

It can be readily verified that all the conditions on $\mathcal{P}$ used in this section are satisfied by standard tessellations such as the familiar cubic tessellation.

It is easy to give examples in which removal of P preserves global topology but not local topology in the neighborhood of P. A simple example is shown in Figure 2; here $N_{\mathcal{P}}(P)$ has no tunnel but $N_{\mathcal{P}}^*(P)$ has a tunnel, but globally, a tunnel exists both before and after P is removed.

5 Concluding remarks

We have shown that in an SN set of tiles, if deletion of a tile preserves local topology, then it preserves global topology; thus simple tiles can be characterized locally. This need not be true in a non-SN set of tiles; and even in an SN set, a tile may be simple even if its deletion does not preserve local topology. It would be of interest to find good characterizations of simple sets of tiles, i.e., sets of tiles whose simultaneous deletion preserves global topology, at least in the SN case.

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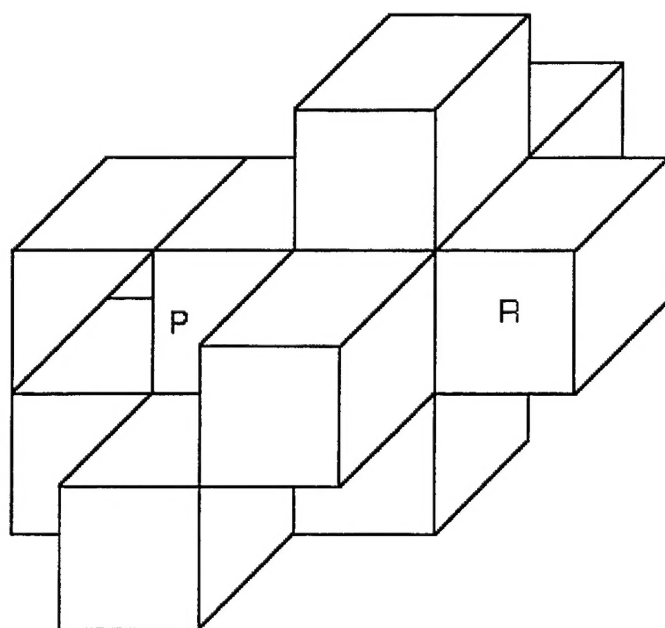
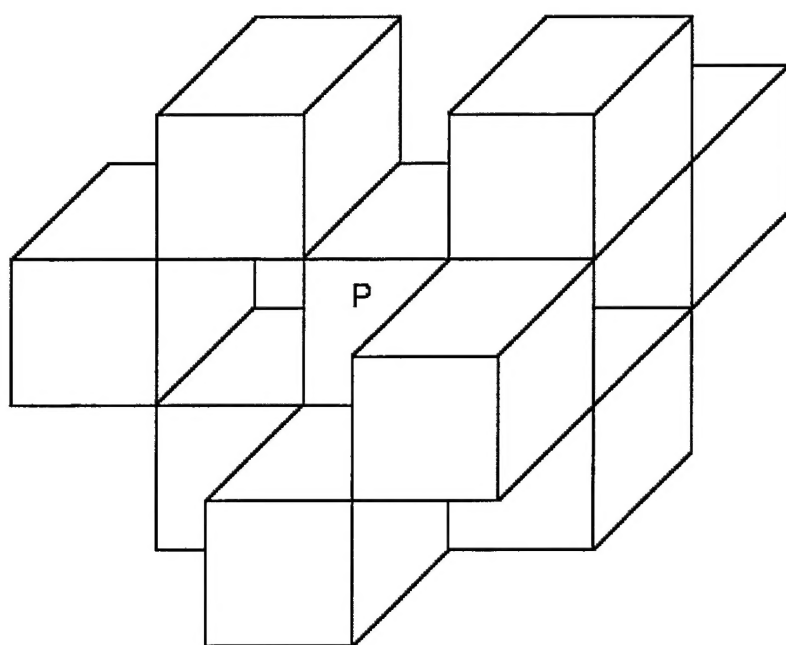Figure 1: Removal of P preserves local topology but not global topology.

Figure 2: Removal of P preserves global topology but not local topology.

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13. ABSTRACT (Maximum 200 words)

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